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ON MEAN ELEMENTS FOR SATELLITE ORBITS PERTURBED BY THE ZONAL HARMONICS OF THE GEOPOTENTIAL

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Users of orbital elements are often puzzled by the different ways in which mean elements are defined. This applies to all the standard elements, but the difficulty is most pronounced in regard to semi-major axis and mean motion.

This paper relates mean elements to the perturbations of a satellite caused by the Earth's zonal harmonic J_2, and shows how elements can be defined so that perturbations in spherical coordinates take an especially simple form. The paper concentrates on the first-order effects of J_2, with formulae given to connect the preferred elements to the elements of Kozai, but it also considers the extension to second order in J_2 and first order in the general zonal harmonic.

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ABSTRACT

Users of orbital elements are often puzzled by the different ways in which mean elements are defined. This applies to all the standard elements, but the difficulty is most pronounced in regard to semi-major axis and mean motion.

This paper relates mean elements to the perturbations of a satellite caused by the Earth's zonal harmonic J_2, and shows how elements can be defined so that perturbations in spherical coordinates take an especially simple form. The paper concentrates on the first-order effects of J_2, with formulae given to connect the preferred elements to the elements of Kozai, but it also considers the extension to second order in J_2 and first order in the general zonal harmonic.

1 INTRODUCTION

It is a notorious fact that the orbital elements derived by the computer programs of different institutions are likely to be defined in different ways, and this is a frequent cause of confusion to users. The difficulty is an obvious one when entirely different element sets are used, but it is just as serious when the element sets are in principle the same. In the present paper we assume the most commonly used 'standard' set, in which a, e, i, Ω, ω and M are supplemented by n, the mean motion. If osculating elements were published, there would be no difficulty, n not even being necessary if the value of μ (= GM) is known; in practice, however, it is always 'mean' elements that are published and it is the precise meaning of these that causes the problem.

If ζ is any member of the element set, the relation between osculating ζ and mean $\bar{\zeta}$ is,

$$\zeta = \bar{\zeta} + \delta\zeta \; , \tag{1}$$

where $\delta\zeta$ symbolizes the combination of all short-period perturbations, whilst $\bar{\zeta}$ has long-term variation only. It is immediately apparent that the split is not uniquely defined, since the origin of $\bar{\zeta}$ is essentially arbitrary - thus any quasi-constant component of $\bar{\zeta}$, in a given split, can be transferred from $\bar{\zeta}$ to $\delta\zeta$ without changing the nature of the split, so long as the order of magnitude of this component is no greater than that inherent in $\delta\zeta$.

We are supposing here that the only perturbations are due to the zonal harmonics, J_ℓ, and will be mainly concerned with J_2. Then the variation of $\bar{\zeta}$ is assumed to be made up of the usual secular and long-period terms. Together with the terms of $\delta\zeta$, these can be obtained by integration of Lagrange's planetary equations. The variation of the fast element, M, is obtained via introduction of the subsidiary element, σ, defined so that

$$M = \int_0^t n\,dt + \sigma \qquad \text{and} \qquad \dot{M} = n + \dot{\sigma} \; . \tag{2}$$

Direct integration of Lagrange's equations, with t (or in practice M) as integration variable, involves the truncation of power series in e; if v (true anomaly) is used, on the other hand, a complete (untruncated) solution can be derived for the first-order variation of ζ due to J_ℓ, and likewise /1/ for the second-order variation due to J_2. But these two different techniques for the integration immediately lead to alternative definitions of $\bar{\zeta}$, on the basis of a hypothetical further integration with respect to M and v respectively; if this integration is taken over a complete revolution, it leads to

the concept of a 'θ-mean' element (where θ is M or v) such that the θ-mean value of $\delta\zeta$, relative to the θ-mean $\bar\zeta$, is zero.

The use of M-mean elements, as in /2/ and /3/, is superficially natural, but the complexity and the inaccuracy associated with the e-truncation are unattractive. It would be possible just to adopt v-mean elements instead, but two further considerations enter the picture. First, there might be an advantage if a particular derived quantity, viz r (the radial distance), had as small a bias as possible relative to its value, $\bar r$, obtained from the $\bar\zeta$; this was evidently a factor in the definition of the widely used elements of Kozai /4/. Second, a definition of the $\bar\zeta$ would be especially attractive if the associated $\delta\zeta$ could be combined into a triple of very compact expressions for perturbations in coordinates. The latter consideration motivated the study of the present author /1/, based on a particular system of spherical-polar coordinates and on earlier work /5,6/.

In the current paper the use of $\bar\zeta$ will normally refer to the mean elements of /1/, so that $\bar a$ will be identical with the quantity a' to be introduced in the next section. These $\bar\zeta$ will be collated with the mean elements of Kozai, denoted by ζ_K for distinction, since the wide usage of the latter includes their use in the RAE orbit-determination program PROP /7/.

2 MEAN SEMI-MAJOR AXIS AND MEAN MEAN MOTION

Since $n^2a^3 = \mu$, it might seem that the definition of $\bar n$ should be automatically tied to $\bar a$, being such that $\bar n^2\bar a^3 = \mu$. There are advantages in making $\bar n$ and $\bar a$ independent, however, with $\bar n^2\bar a^3$ equal to a modified μ. This was recognized by Kozai /4/, whose version of Kepler's third law may be written as $\bar n^2\bar a^3 = \mu(1 - \bar K\bar q\bar h)$, where

$$K = \frac{3}{2}J_2\left(\frac{R}{p}\right)^2, \qquad p = aq^2 = a(1-e^2), \qquad h = 1 - \frac{3}{2}f, \qquad f = \sin^2 i, \tag{3}$$

and R is the Earth's equatorial radius. Ironically, though, we shall find that to first order in J_2 the preferred definitions of $\bar n$ and $\bar a$ satisfy the unmodified law; taken to second order in J_2, and with the general even harmonic allowed for to first order, the modified law takes the form (summation for even $\ell \geq 4$)

$$\bar n^2\bar a^3 = \mu\left\{1 - \frac{1}{24}\bar K^2\bar q^3(8 - 8\bar f - 5\bar f^2) + 2\bar q^3\sum_\ell \gamma_\ell J_\ell\left(\frac{R}{\bar p}\right)^\ell A_\ell(\bar i)\, Q_\ell(\bar e)\right\}, \tag{4}$$

where the constants (γ_ℓ) and functions (A_ℓ and Q_ℓ) will be specified later.

In seeking a compact formula for δr, r being one of the spherical coordinates referred to in the Introduction, we have to deal with the term $(\partial r/\partial a)\,\delta a$, where

$$\frac{\partial r}{\partial a} = \frac{r}{a} = \frac{q^2}{1 + e\cos v} \qquad \left(\text{since } \frac{p}{r} = 1 + e\cos v\right); \tag{5}$$

thus we would like $1 + \bar e\cos\bar v$ to be a factor of δa, to cancel the denominator in (5). Following the earlier work /5,6/, we note that conservation of energy (for the zonal potential field) yields an exact constant a' (to whatever order the analysis is taken), where

$$\frac{1}{a'} = \frac{1}{a} + \frac{2U}{\mu}, \qquad \text{where } U = \sum_\ell U_\ell \quad \text{and} \quad U_\ell = -\frac{\mu}{r}J_\ell\left(\frac{R}{r}\right)^\ell P_\ell(\sin\beta), \tag{6}$$

U being the disturbing potential, β the latitude, and P_ℓ the usual Legendre polynomial. It follows that $a - a' = 2aa'U/\mu$ and hence contains $(R/r)^2$ as a factor (from every U, since $\ell \geq 2$). But $R/r = (R/p)(1 + e\cos v)$, so if $\delta a = a - a'$ the required cancelling takes place. This is why a' is the preferred $\bar a$.

There is no corresponding consideration for $\bar n$, and the obvious course is to identify $\bar n$ with the secular component of $\dot M$; this makes it only weakly dependent on the definition of $\bar M$ itself, as $\dot{\bar M}$ is independent of any arbitrary constant in $\bar M$. We work from (2), but

need to eliminate the variability of n , which we can do via n' , defined such that $n'^2 a'^3 = \mu$. Thus (6) gives

$$n = n' - 3naU/\mu + O(U^2) \,, \qquad \text{whence} \quad \bar{n} = \dot{\bar{M}} = n' + \dot{\sigma} - 3\bar{n}\bar{a}\bar{U}/\mu \,, \tag{7}$$

where, since $\dot{\bar{M}}$ and $\dot{\sigma}$ are the secular components of the $\bar{M}$ and σ variation, $\bar{U}$ has to be interpreted as the M-mean value of U . To derive the γ_ℓ terms of (4), we develop $\bar{n}$, in (7), in the form $n'\left\{1 + \bar{q}^3 \sum_\ell \gamma_\ell J_\ell (R/p)^\ell A_\ell Q_\ell\right\}$, but before pursuing this we look more closely at $\bar{n}$ and $\bar{a}$ for the J_2-only field.

J_2 analysis

Substituting $\frac{1}{2}(3f\sin^2 u - 1)$ for $P_2(\sin\beta)$ in (6), where $u = v + \omega$, we get

$$\bar{U}_2 = \frac{\mu}{12\pi}\frac{\bar{K}}{\bar{p}}\int_0^{2\pi}\left(\frac{p}{r}\right)^3 (2h + 3f\cos 2u)\,dM \,. \tag{8}$$

Since $(p/r)^2\, dM/dv = q^3$, integration gives $\bar{U}_2 = \frac{1}{3}\mu(\bar{K}/\bar{p})\bar{q}^3\bar{h}$ to first order in J_2 . Taking $\dot{\sigma} = \bar{K}\bar{n}\bar{q}\bar{h}$ /6/, we have $3\bar{n}\bar{a}\bar{U}/\mu = \dot{\sigma}$, so (7) gives $\bar{n} = n'$.

Thus the preferred values for $\bar{n}$ and $\bar{a}$ are n' (for $\ell = 2$ only) and a' (universally), with $\bar{n}^2\bar{a}^3$ therefore equal to unmodified μ . Kozai, on the other hand, though using the same mean n , has a different mean a , denoted here by a_K and such that $a_K = a'(1 - \frac{1}{3}\bar{K}\bar{q}\bar{h})$; this accounts for his version of Kepler's third law. As Kozai does not give a rationale for a_K , whilst a considerable literature follows his paper /4/ without clarifying the matter, it is worth offering an explanation here.

The attraction of an unbiased δr was noted in the Introduction, and Kozai's δr is unbiased for circular orbits. Though his δr expression (for general e) is more complicated than the one to be given here (equation (18)), the δr of (18) is biased, even for circular orbits, unless $\bar{h} = 0$ ($i = 54.7°$ or $125.3°$). The lack of bias if a_K is used means that, for an orbit that is equatorial as well as circular, r (which now has a fixed value) actually identifies with a_K (neglecting $J_2{}^2$). The value of a (osculating, and also fixed) is different, however, being equal to a_0 , *defined in the next paragraph*. Whilst considering orbits with $e = \sin i = 0$, it is worth remarking on a third candidate (in addition to n' and the unbiased n_0) for $\bar{n}$, viz the rate of change of longitude, n_λ say. Since $\dot{\bar{\omega}} \pm \dot{\bar{\Omega}}$ (sign given by $\cos i$) is $\bar{K}\bar{n}$, under *these* circumstances, we have $n_\lambda = n'(1 + \bar{K})$, so $n_\lambda{}^2 r^3 = \mu(1 + \bar{K})$, and this is of particular interest for a geostationary orbit, where n_λ is equal to the Earth's rotation rate; if a_λ is defined such that $n_\lambda{}^2 a_\lambda{}^3 = \mu$, it will be noted that

$$a_0 - a' = 2(a' - a_K) = 2(a_K - a_\lambda) \,, \tag{9}$$

the 'unit value' of $a_K - a_\lambda$ being about 0.52 km (geostationary orbit assumed).

The quantities n_0 and a_0 , just referred to, are simply the M-mean *values* of n and a , but were introduced (with the zero-suffix notation) by Kozai (in contradistinction to n' and a_K , notated as $\bar{n}$ and $\bar{a}$) in a rather misleading way, since he describes them as 'unperturbed values at epoch' though they are actually *unbiased values that are* independent of epoch (the word 'unperturbed' seems meaningless in the context); unfortunately, this inapposite description is followed in a number of textbooks. The relations between (i) n_0 and (n_K =) n', and (ii) a_0, a_K and a' , are:

$$\text{(i)} \quad n_0 = n'(1 - \bar{K}\bar{q}\bar{h}) \,; \qquad \text{(ii)} \quad a_0 = a_K(1 + \bar{K}\bar{q}\bar{h}) = a'(1 + \tfrac{2}{3}\bar{K}\bar{q}\bar{h}) \,. \tag{10}$$

Analysis for J_ℓ ($\ell > 4$)

General analysis /5/ yields expressions for $\dot{\sigma}$ (secular) due to J_ℓ , *together* with the

corresponding $\bar{U}_\ell$, for substitution in (7). On combining the two terms, for each ℓ , we get

$$\bar{n} = n'\left\{1 + \sum_\ell J_\ell \left(\frac{R}{p}\right)^\ell \nu_\ell\right\} , \quad \text{where } \nu_\ell = -C_\ell A_\ell(\bar{I})\bar{q}^3\bar{e}^{-1}dB_\ell(\bar{e})/d\bar{e} \; ; \tag{11}$$

here $A_\ell(I)$ is a polynomial in f (if ℓ is even) and $B_\ell(e)$ is a polynomial in e^2 , normalized such that $A_\ell(0) = B_\ell(0) = 1$, with C_ℓ a pure number, given by $C_2 = \frac{1}{2}$ and $C_\ell/C_{\ell-2} = -(\ell - 1)/\ell$ if $\ell \geq 4$. (A_ℓ, B_ℓ and C_ℓ are particular cases of the A_ℓ^k, B_ℓ^k and C_ℓ^k defined in /5/, k being zero, recurrence relations having been given for A and B as well as C .) The derivative of $B_\ell(e)$ has leading term $\frac{1}{2}(\ell - 1)(\ell - 2)$, so we re-normalize (11), writing $\nu_\ell = \gamma_\ell\bar{q}^3A_\ell(\bar{I})Q_\ell(\bar{e})$, where

$$Q_\ell(e) = \left[e^{-1}dB_\ell(e)/de\right] / \left[\tfrac{1}{2}(\ell - 1)(\ell - 2)\right] \quad \text{and} \quad \gamma_\ell = -\tfrac{1}{2}(\ell - 1)(\ell - 2)C_\ell \; . \tag{12}$$

The recurrence relation for $Q_\ell(e)$, using odd values of ℓ as well as even ones, is

$$Q_\ell = \left[(2\ell - 5)Q_{\ell-1} - (\ell - 4)q^2Q_{\ell-2}\right] / (\ell - 1) \; , \tag{13}$$

for $\ell \geq 4$, with $Q_2 = 0$ and $Q_3 = 1$. The recurrence relation for γ_ℓ , using even values of ℓ only, is (for $\ell \geq 6$, with $\gamma_4 = 9/8$)

$$\gamma_\ell = -\left[(\ell - 1)^2(\ell - 2)/\ell(\ell - 3)(\ell - 4) \; \gamma_{\ell-2}\right] . \tag{14}$$

Since J_ℓ (for $\ell \geq 4$ and even) contributes the factor $1 + 2J_\ell(R/p)^\ell\nu_\ell$ to $\bar{n}^2\bar{a}^3/\mu$ (with $\bar{a} = a'$), we can write (on evaluating the γ_ℓ as far as $\ell = 8$)

$$\bar{n}^2\bar{a}^3 = \mu\left[1 + \bar{q}^3\left\{\frac{9}{4}J_4\left(\frac{R}{\bar{p}}\right)^4 A_4Q_4 - \frac{25}{4}J_6\left(\frac{R}{\bar{p}}\right)^6 A_6Q_6 + \frac{735}{64}J_8\left(\frac{R}{\bar{p}}\right)^8 A_8Q_8 - \dots - \frac{1}{24}\bar{K}^2(8 - 8\bar{f} - 5\bar{f}^2)\right\}\right] . \tag{15}$$

(The $\bar{K}^2$ term is taken from /1/, where it is not derived. A full derivation should eventually be published /8/.)

3 MEAN ELEMENTS AND COMPLETE FIRST-ORDER SHORT-PERIOD PERTURBATIONS ASSOCIATED WITH J_2

We have seen that the expressions, in coordinates, for short-period perturbations depend on the definitions of mean elements. In /6/ the author gave expressions for the first-order perturbations due to an arbitrary harmonic (tesserals included), using a particular system of cylindrical-polar coordinates; complete expressions were given for J_2 perturbations, but for the general harmonic (and, additionally, for $J_2{}^2$ perturbations) they were e-truncated, effectively applying only to near-circular orbits. More recently, the results of a complete analysis, to second order in J_2 and first order in J_3 , have been summarized /1/, and it is intended that full details will follow /8/. The analysis showed that a spherical-polar system is actually preferable to the cylindrical-polar system, and that J_2-based definitions of two of the mean elements ($\bar{a}$ and $\bar{e}$) had not been optimal in the original study - this conclusion was anticipated by the 'Note added in proof' at the end of /6/. Having considered here the relation of the optimal $\bar{a}$ and $\bar{n}$ to Kozai's elements, we now do the same for the remaining elements, concluding with the resulting expressions for first-order perturbations in coordinates.

The optimal $\bar{\Omega}$, $\bar{\omega}$ and $\bar{M}$ are identical with the elements of Kozai, so we only have to

consider $\bar{e}$ and $\bar{I}$, but it is also worth covering the derived element $\bar{p}$. In terms of the optimal (preferred) $\bar{e}$ and $\bar{p}$, Kozai's elements are given by

$$e_K = \bar{e}\left[1 - \tfrac{1}{3}\bar{K}\bar{h}\bar{q}^2/(1 + \bar{q})\right] \qquad \text{and} \qquad p_K = \bar{p}\left[1 + \tfrac{1}{3}\bar{K}\bar{h}(2 - 3\bar{q})\right] . \tag{16}$$

The reward for using our preferred $\bar{a}$ (= a') and $\bar{e}$ is to obtain the very simple expressions for the complete perturbations δr and δw that follow. Here the spherical-coordinate system (r, b, w) is based on the instantaneous position of the mean orbital plane, defined by $\bar{I}$ and $\bar{\Omega}$, such that w is the in-plane angle (quasi-longitude) and b is the out-of-plane angle (quasi-latitude). For the simplest possible expression for δb (where $\bar{b}$ is automatically zero by definition of the reference plane), the $\bar{I}$ we need is within $O(\bar{K}\bar{e})$ of the extreme values of β (at the apexes of the orbit). Kozai's inclination, on the other hand, is within $O(\bar{K}\bar{e}^2)$ of the ~~M-mean~~ i . This seems a very much less natural choice (and is a notable example of the allure of the intuitive meaning of 'mean'); it differs from our preferred $\bar{I}$ by an $O(\bar{K})$ quantity, since

$$i_K = \bar{I} + \tfrac{1}{4}\bar{K} \sin 2\bar{I} \ . \tag{17}$$

The mean elements, to order J_2 , have now all been established, and the complete (first-order) expressions for coordinate perturbations can be quoted. They are:

$$\left.\begin{aligned} \delta r &= \tfrac{1}{6}\bar{K}\bar{p}(\bar{f}\cos 2\bar{u} - 2\bar{h}) , \qquad \delta b = \tfrac{1}{6}\bar{K}\bar{e}\sin 2\bar{I}\,\{\sin(\bar{u} + \bar{v}) - 3\sin\bar{\omega}\} \\ \text{and} \\ \delta w &= \tfrac{1}{12}\bar{K}\{\bar{f}\sin 2\bar{u} + 4\bar{e}\bar{f}\sin(\bar{u} + \bar{\omega}) + 8\bar{e}\bar{h}\sin\bar{v}\} \end{aligned}\right\} . \tag{18}$$

The corresponding second-order perturbations are not given here (see /1/) but it is worth noting that the number of terms required for δr, δb and δw are 6, 7 and 11 respectively. This is in striking contrast to the number of terms required for the six osculating elements, which is 140 overall.

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